Learning about Square Roots: A common Core Trajectory
By Cornelis (Kees) de Groot

The problem:

I noticed many teachers demonstrating the solution to $x^2 = 36$, for example, as follows:

> We must do the same thing to both sides of the equation, so we will take the square root on each side:
> $\sqrt{x^2} = \sqrt{36}$
> The square root of x^2 equals *x* and the square root of 36 equals plus or minus 6.
> $x = \pm 6$

The teacher makes two mistakes, but arrives at the correct answer. The first mistake is that $\sqrt{x^2} \neq x$, but rather $\sqrt{x^2} = |x|$.
The same must hold for $\sqrt{36}$, which we can see as $\sqrt{6^2}$.
By definition of the **principal** square root, the $\sqrt{36}$ has to be 6.
There for the equation needs to be solved as follows:
$x^2 = 36$
$\sqrt{x^2} = \sqrt{36}$
$|x| = 6$
$x = \pm 6$

The reason why many people think that the square root of 36 equals plus or minus 6 is related to confusion from the following type of definition:

> a **square root** of a number *x* is a number *r* such that $r^2 = x$, or, in other words, a number *r* whose *square* (the result of multiplying the number by itself, or $r \times r$) is *x*. For example, 4 is a square root of 16 because $4^2 = 16$. (source: Wikipedia http://en.wikipedia.org/wiki/Square_root)

Because $(-4)^2 = 16$ as well, many people think that therefore the square root of 16 can be -4 as well. This is incorrect and is confounded by trying to solve the above equation incorrectly, but with correct answers.

Common Core State Standards for Mathematics

Grade 8

The Number System 8.NS
Know that there are numbers that are not rational, and approximate them by rational numbers.
1. Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion, which repeats eventually into a rational number.
2. Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π2). For example, by truncating the decimal expansion of √2, show that √2 is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations.

Expressions and Equations 8.EE
Work with radicals and integer exponents.
2. Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that √2 is irrational.

Geometry 8.G
Understand and apply the Pythagorean Theorem.
6. Explain a proof of the Pythagorean Theorem and its converse.
7. Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.
8. Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.

High School

Number and Quantity
The Real Number System N -RN
Extend the properties of exponents to rational exponents.
1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. For example, we define $5^{\frac{1}{3}}$ to be the cube root of 5 because we want $(5^{\frac{1}{3}})^3 = (5^3)^{\frac{1}{3}}$ to hold, so $(5^{\frac{1}{3}})^3$ must equal 5.
2. Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Use properties of rational and irrational numbers.
3. Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Algebra
Reasoning with Equations and Inequalities A -RE I
Understand solving equations as a process of reasoning and explain the reasoning
1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.
2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise. Solve equations and inequalities in one variable
4. Solve quadratic equations in one variable.
a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x-p)^2=q$ that has the same solutions. Derive the quadratic formula from this form.
b. Solve quadratic equations by inspection (e.g., for $x^2=49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as *a ± bi* for real numbers *a* and *b*.

Exemplars and the Common Core State Standards

The CCSS for Mathematical Practice are comprised of the following:	*Exemplars* rubric criteria from the "Practitioner Level" supports CCSS by requiring students to do the following in order to meet the standard:
MAKE SENSE OF PROBLEMS AND PERSEVERE IN SOLVING THEM.	<u>Problem Solving</u> • A correct strategy is chosen based on mathematical situation in the task. • Evidence of solidifying prior knowledge and applying it to the problem-solving situation is present. • Planning or monitoring of a strategy is evident <u>Reasoning and Proof</u> • A systematic approach and/or justification of correct reasoning is present. This may lead to: ○ clarification of the task. ○ exploration of mathematical phenomenon. <u>Representations</u> • Appropriate and accurate mathematical representations are constructed and refined to solve problems or portray solutions.
REASON ABSRACTLY AND QUANTITATIVELY.	<u>Reasoning and Proof</u> • Arguments are constructed with adequate mathematical basis. • A systematic approach and/or justification of correct reasoning is present. This may lead to: ○ clarification of the task. ○ exploration of mathematical phenomenon. <u>Representations</u> • Appropriate and accurate mathematical representations are constructed and refined to solve problems or portray solutions. <u>Communication</u> • Formal math language is used throughout the solution to share and clarify ideas.
CONSTRUCT VIABLE ARGUMENTS AND CRITIQUE THE REASONING OF OTHERS.	<u>Problem Solving</u> • Evidence of solidifying prior knowledge and applying it to the problem-solving situation is present. <u>Reasoning and Proof</u> • Arguments are constructed with adequate mathematical basis. • A systematic approach and/or justification of correct reasoning are/is present. ○ Exploration of mathematical phenomenon. <u>Communications</u> • A sense of audience or purpose is communicated. • Communication of an approach is evident through a methodical, organized, coherent sequenced and labeled response. <u>Representations</u> • Appropriate and accurate mathematical representations are constructed and refined to solve problems or portray solutions.

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MODEL WITH MATHEMATICS	<u>Problem Solving</u> • Evidence of solidifying prior knowledge and applying it to the problem-solving situation is present. • Planning or monitoring of strategy is evident. <u>Reasoning and Proof</u> • Arguments are constructed with adequate mathematical basis. • A systematic approach and/or justification of correct reasoning are/is present. <u>Representations</u> • Appropriate and accurate mathematical representations are constructed and refined to solve problems or portray solutions. <u>Communication</u> • Formal math language is used throughout the solution to share and clarify ideas.
USE APPROPRIATE TOOLS STRATEGICIALLY	<u>Problem Solving</u> • A correct strategy is chosen based on mathematical situation in the task. • Evidence of solidifying prior knowledge and applying it to the problem-solving situation is present. • Planning or monitoring of strategy is evident.
ATTEND TO PRECISION.	<u>Problem Solving</u> • The Practitioner must achieve a correct answer. <u>Representations</u> • Appropriate and accurate mathematical representations are constructed and refined to solve problems or portray solutions. <u>Communications</u> • A sense of audience or purpose is communicated. • Communication of an approach is evident through a methodical, organized, coherent sequenced and labeled response • Formal math language is used throughout the solution to share and clarify ideas.
LOOK FOR AND MAKE USE OF STRUCTURE.	<u>Problem Solving</u> • Planning or monitoring of strategy is evident. <u>Reasoning and Proof</u> • Exploration of mathematical phenomenon. • Noting patterns, structures and regularities. <u>Connections</u> • Mathematical connections or observations are recognized.
LOOK FOR AND EXPRESS REGULARITY IN REPEATED REASONING.	<u>Problem Solving</u> • Planning or monitoring of strategy is evident. <u>Reasoning and Proof</u> • Noting patterns, structures and regularities. <u>Connections</u> • Mathematical connections or observations are recognized.

The Trajectory

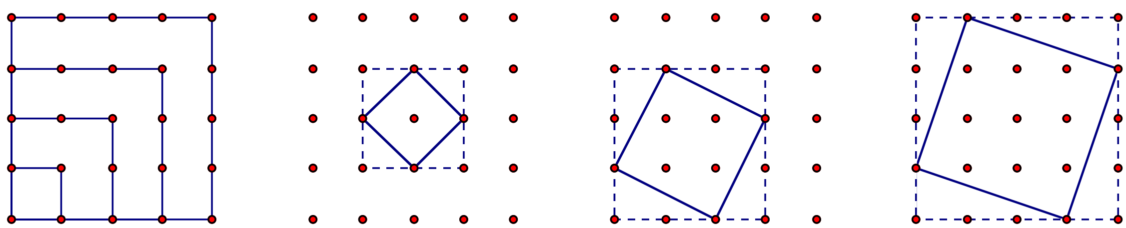
1. <u>Representing the square root as the **measure** of a length of the side of a square</u>

> 1.1 Using a Geoboard, create squares of areas 1, 4, 9, and 16 and determine the length of the sides of each square.

This exercise establishes that given an area of a square we can find the length of its sides. Since length is always a quantity greater or equal to 0, this work sets the stage for the concept of the principal square root. Also, in this context an area is a quantity greater than or equal to 0. This in turn helps students see that a principal square root of a negative quantity cannot exist (in the real number system).

So we can say that the length of a side of the square with area 16 has to be 4, because $4 \times 4 = 16$ or $4^2 = 16$

> 1.2 Next, as a challenge, use the Geoboard to construct squares with area 2, 10, and 13.



There is not an immediate obvious geometric and numeric way to solve this. But we can articulate that we are looking for a positive number (being the measure of the length) that when multiplied by itself yields the area.

number	square
1.5	2.25
1.4	1.96
1.45	2.1025
1.425	2.030625
1.42	2.0164
1.415	2.002225
1.414	1.999396
1.4145	2.00081025
1.4143	2.00024449
1.4142	1.99996164
1.41421	1.999989924
1.414215	2.000004066

1.3 We can begin with estimating the length of a square with area 2 by noticing that it is in between 1 and 4 (perfect squares). Thus the length of the side must be between 1 and 2. We can get closer and closer to the length by testing with the calculator. After trying this for the three examples above we can invite students to use the $\sqrt{\ }$ button on their calculator and notice that this does exactly what they did through guess and check.

In this manner we use a geometric context to establish the concept of the principal square root to represent the measure of the length of the side of a square with a given area. From this we can derive that if $\sqrt{A}$

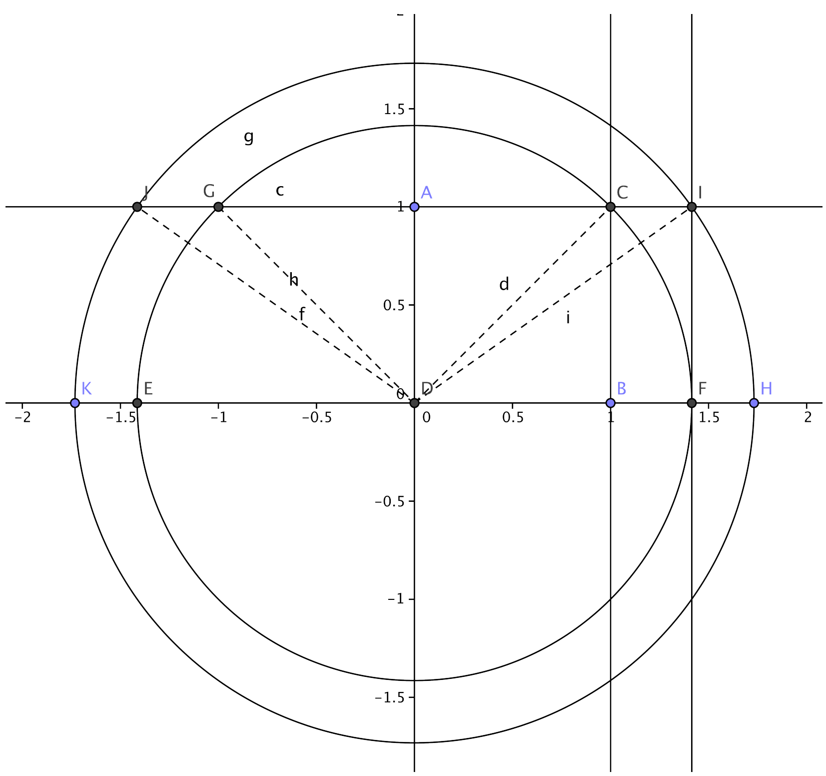
represents the measure of the length of the sides of a square with area A, $\sqrt{A}\times\sqrt{A}=A$ or $(\sqrt{A})^2=A$ as in $s^2=A$.

2. Working with square roots as numbers or quantities (mathematical objects).

In section 1.3 above it can be tempting to think that we are looking to solve $x^2=2$ and $x^2=10$ and $x^2=13$. This only works if we are clear about a domain and range restriction that is necessitated by the context of area and length. To start working with radicals as numbers, and later as functions, we need to move to different representations, namely, we will consider radicals in their relation to number systems as well as representing a location on the number line.

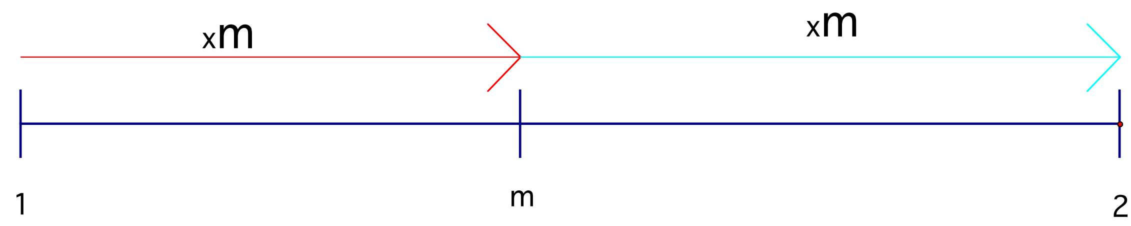
2.1 Through using a calculator's (or spreadsheet's) capacity students can investigate that square roots can be classified as terminating decimal fractions, non-terminating repeating decimal fractions, or non-terminating non-repeating decimal fractions. The first two representing rational numbers and the last irrational numbers.

2.2 Students can construct locations of square roots of whole number radicands on the number line as follows (this requires their understanding of the Pythagorean theorem).



In the figure above one can see that the construction yields the location of $\sqrt{2}$ and $\sqrt{3}$ as well as $-\sqrt{2}$ and $-\sqrt{3}$ by symmetry.

2.3 Next we will look at a square root as a **geometric mean**. We will again use the number line to establish this conceptual understanding of, in this case, $\sqrt{2}$.

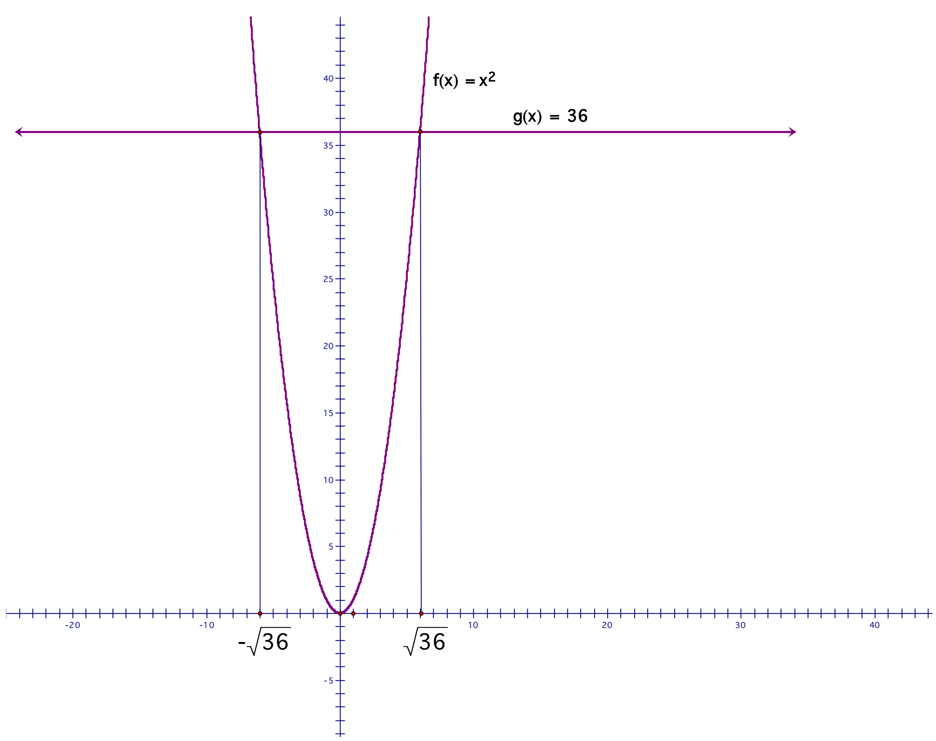


Let's say we think that the $\sqrt{2}$ is located in *m*. Point *m* is then located such that we need to first multiply 1 by *m* to get to that location and then we need to multiply *m* by itself to get to the location 2. Thus we can see that $m^2 = 2$, and therefore *m* had to be the location for $\sqrt{2}$. We will return to this idea later as we deal with rational exponents.

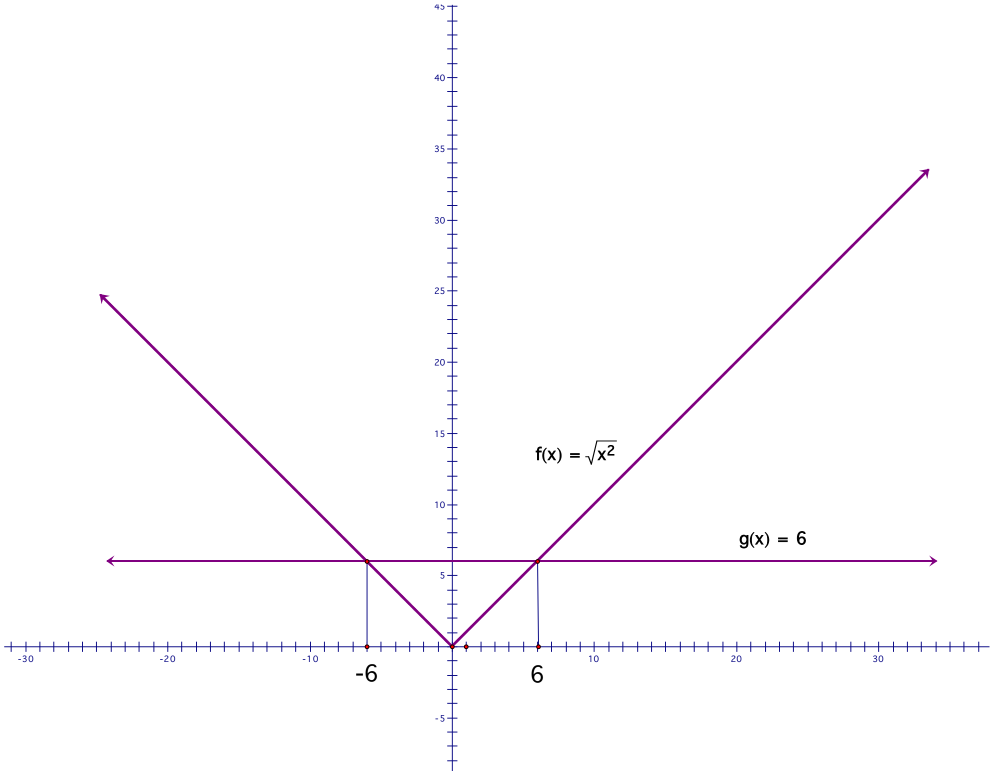
3. Working with radical functions

We will return to the initial problem where we were trying to solve a quadratic equation such as $x^2 = 36$. We can regard this as a question that asks us to determine where the function values of $f(x) = x^2$ and $g(x) = 36$ are the same.

3.1 Graphically this means we try to determine at what values of the domain (input values) the two functions intersect (have the same output values). By the symmetry of $f(x) = x^2$ this will yield two results.



3.2 We can also investigate the equivalent equation $\sqrt{x^2} = \sqrt{36}$ graphically. To achieve this we need to graph $f(x) = \sqrt{x^2}$ and $g(x) = \sqrt{36} = 6$. The result is as follows:



Therefore solving $\sqrt{x^2} = \sqrt{36}$ is equivalent to solving $|x| = 6$.

4. <u>Taking an algebraic approach.</u>

In algebra we often express equations in a several forms. Expressing a quadratic equation in the standard for has the advantage that one can use factoring the expression in two linear factors and solve these as follows:

$$x^2 = 36$$
$$x^2 - 36 = 0$$
$$(x-6)(x+6) = 0$$
$$x - 6 = 0 \text{ or } x + 6 = 0$$
$$x = 6 \text{ or } x = -6$$

This method is more likely to prevent students to give only one possible answer for the original equation.

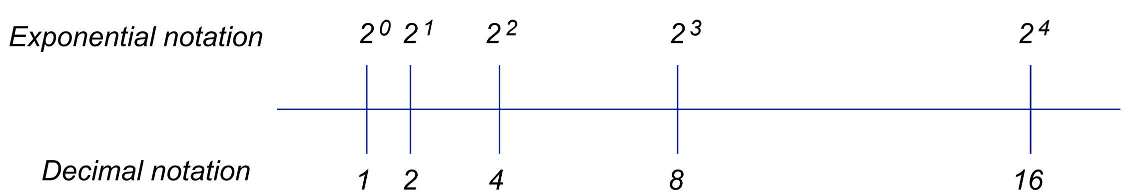
5. Radicals and rational exponents.

We return to using the concept of a number identifying a location on the number line (or a distance from 0).

5.1 Pre-requisite knowledge:

- Students should be familiar with exponential numbers and their real number equivalent for positive and negative exponents.
- Students should know that exponential numbers with a positive base and negative exponents are all located between 0 and 1.
- Students should be familiar with the reciprocal relationship between positive and negative exponents.
- Students should be familiar with multiplying and dividing exponential numbers
- Students should know that an *nth-root* is a solution to a polynomial equation of degree *n*.
- Students should be familiar with locating exponential numbers on the number line.
- Students should be familiar with how exponential numbers are spaced on the number line as contrasted with integer and rational number spacing.
- Students should be familiar with geometric mean and arithmetic mean and how these are different.
- Students should be able to distinguish "in the middle" from "in between."

Figure 1



Notes with figure 1: While exponent 2 is in the middle of exponents 1 and 3, 2^2 is not in the middle of 2^1, and 2^3. Distances (differences) between consecutive exponential numbers grow by the rate of the base. This means the distance (difference) between a^{n+2} and a^{n+1} is *a* times larger than the distance (difference) between a^{n+1} and a^n.

To assess this understanding ask students the following:
Given the difference between 10^{60} and 10^{61} and the difference between 10^{90} and 10^{91}, which of the following is true:

A. $10^{61} - 10^{60} > 10^{91} - 10^{90}$
B. $10^{61} - 10^{60} = 10^{91} - 10^{90}$
C. $10^{61} - 10^{60} < 10^{91} - 10^{90}$

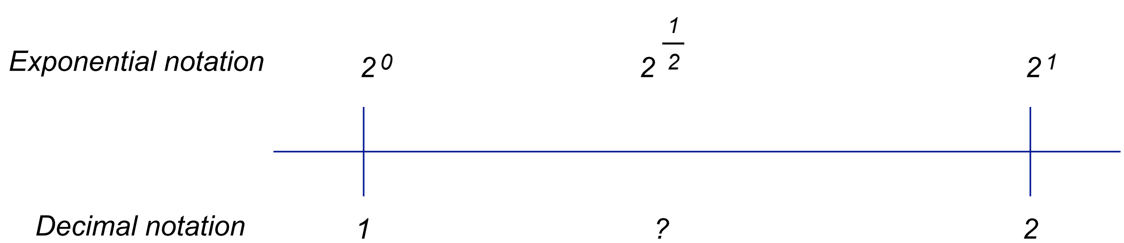
Students who do not understand the effect on spacing of exponential numbers will most often choose B, because they look at the difference of the exponents. You can help these students to think about how many whole numbers are between 10 and 100 (i.e. 90) and 100 and 1000 (900).

5.2 Introducing rational exponents conceptually

We are now going to focus on a section between two consecutive exponential numbers with integer exponents.
First ask students to identify numbers (and their locations) on the real number representation side (below) between 1 and 2.
Then ask students to identify a number between 2^0 and 2^1, using exponential notation. If necessary guide students to realize that this should be a rational exponent. (Developmentally students should realize the number line to be continuous rather than discrete to participate in this line of reasoning. Also think of the completeness theorem from Archimedes.)

Then pose the question to the class to determine where $2^{\frac{1}{2}}$ is located between 2^0 and 2^1 and to find the decimal (real number) equivalent position. (See figure 2 below.)

Figure 2



Many students will assume (as discussed above with integer exponents) that $2^{\frac{1}{2}}$ is in the middle of 2^0 and 2^1. So we need to scaffold toward the geometric mean using figure 1 and then apply this to figure 2 as show in the Figures 3 and 4.

In figure 3 we can observe (determine) how the multipliers (rates or growth factors) relate at the real number side and the equivalent exponential side.
Here we need to point out the rule of how to multiply exponential numbers and how two subsequent multipliers combine to a single multiplier.

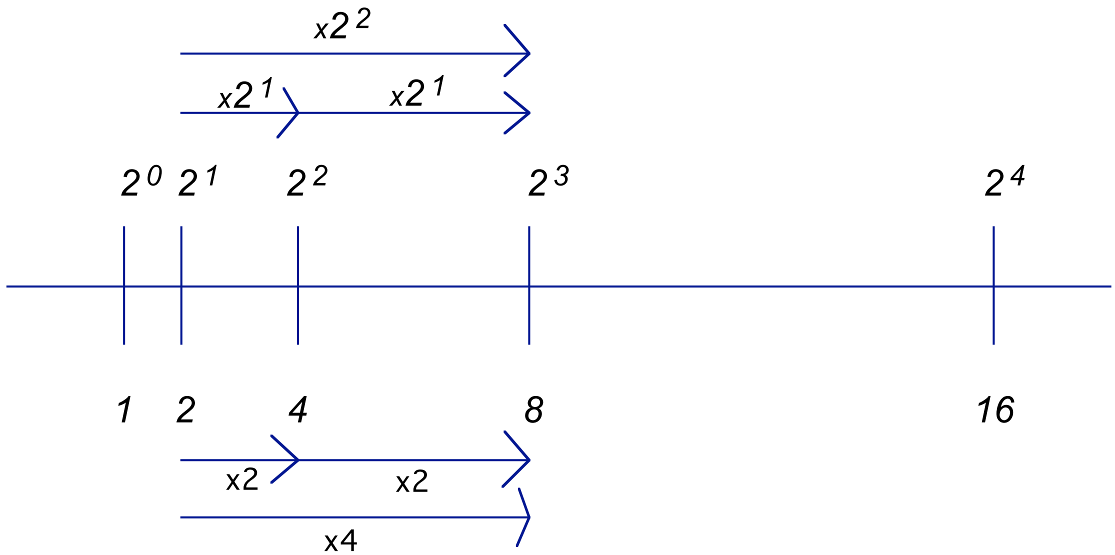
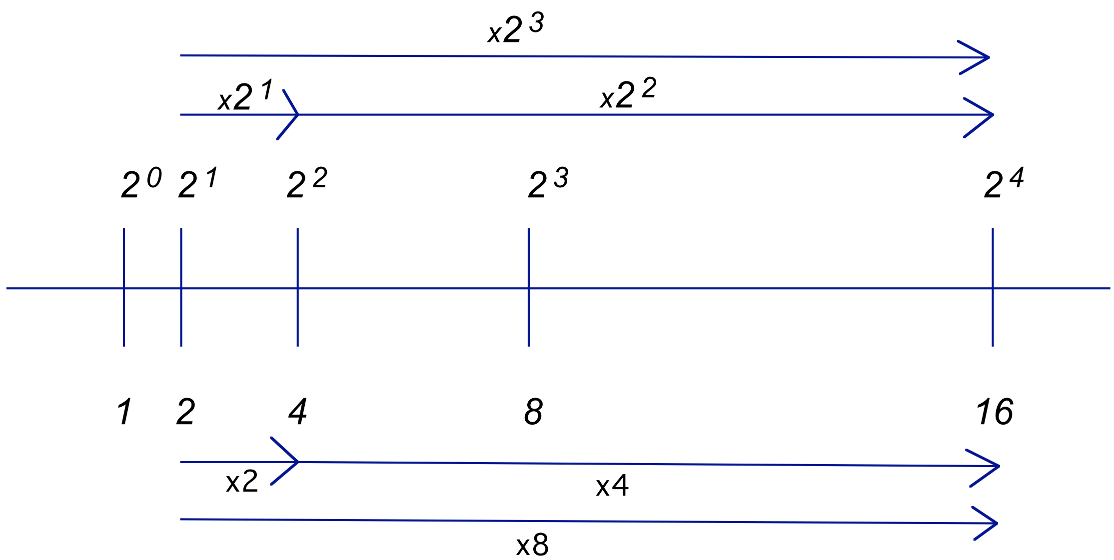
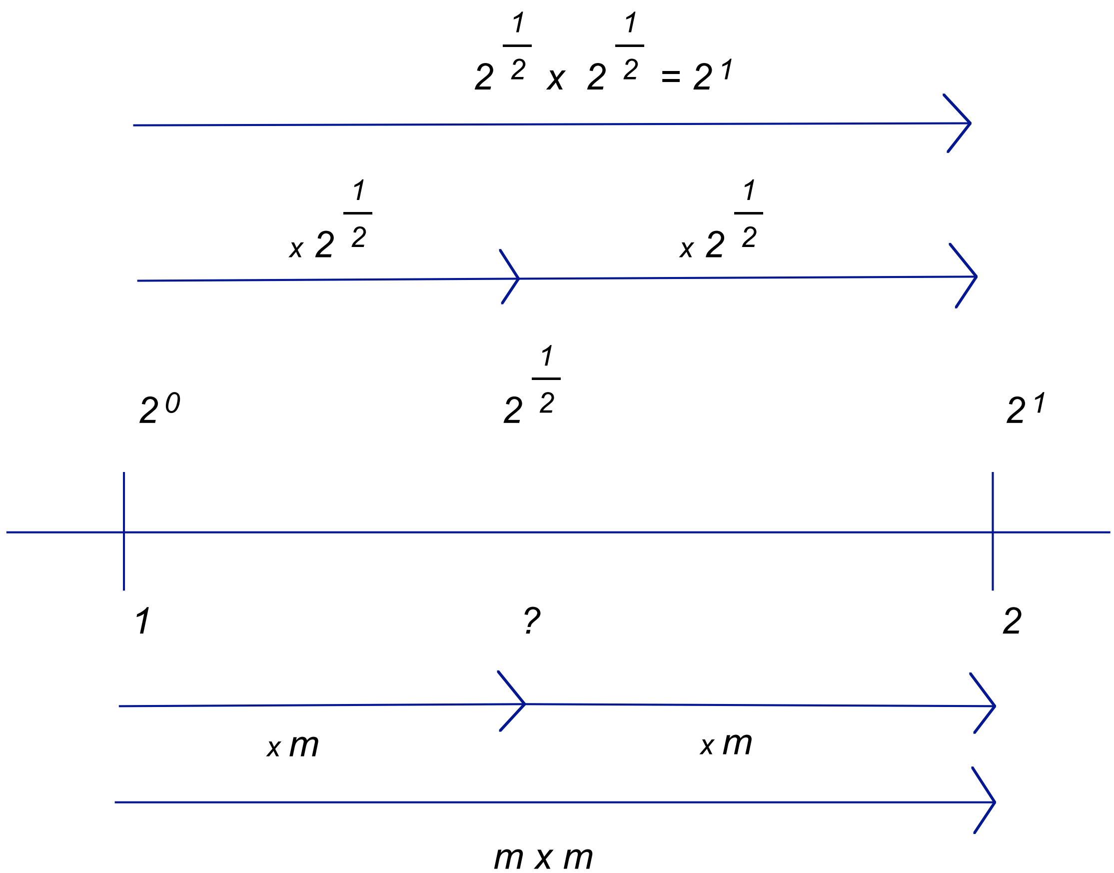


Figure 3
OR



In figure 4 we look at a specific portion of the number line and need to know that the rule for multiplying exponential numbers also works for rational exponents (and even real exponents later). We also need the idea of a geometric mean to determine the equivalent multiplier in the real number context.

Figure 4



From the figure above we can conclude that, if $m = 2^{\frac{1}{2}}$, then *m x m = 2*, which implies that $m^2 = 2$. From this we conclude that $m = \sqrt{2}$.
This then means that $2^{\frac{1}{2}} = \sqrt{2}$

Similarly we can determine that $2^{\frac{1}{3}} = \sqrt[3]{2}$ and so on.

All of the above can be derived in a real world context as well:

Today, in my pond I found 1 square foot of algae. A biologist told me that the amount of algae doubles each day this time of the year.
Possible questions:
1. What area of the pond will be covered with algae in *p* days from now?
(*Positive integer exponents*)
2. What area of the pond was covered with algae *n* days ago?
(*Negative integer exponents*)
3. What area of the pond will be covered with algae1/2 day from now?
(*Positive rational exponents*)
4. What area of the pond was be covered with algae 1/2 days ago?
(*Negative rational exponents*)